

THE DIRICHLET ELLIPTIC PROBLEM INVOLVING REGIONAL FRACTIONAL LAPLACIAN

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Abstract. In this paper, we study the solutions for elliptic equations involving regional fractional Laplacian

$$\begin{cases} (-\Delta)_\Omega^\alpha u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where Ω is a bounded open domain in \mathbb{R}^N ($N \geq 2$) with C^2 boundary $\partial\Omega$, $\alpha \in (\frac{1}{2}, 1)$ and the operator $(-\Delta)_\Omega^\alpha$ denotes the regional fractional Laplacian. We prove that when $g \equiv 0$, problem (1) admits a unique weak solution in the cases that $f \in L^2(\Omega)$, $f \in L^1(\Omega, \rho^\beta dx)$ and $f \in \mathcal{M}(\Omega, \rho^\beta)$, here $\rho(x) = \text{dist}(x, \partial\Omega)$, $\beta = 2\alpha - 1$ and $\mathcal{M}(\Omega, \rho^\beta)$ is a space of all Radon measures ν satisfying $\int_\Omega \rho^\beta d|\nu| < +\infty$. Finally, we provide an Integral by Parts Formula for the classical solution of (1) with general boundary data g .

1. INTRODUCTION

The usual Laplacian operator may be thought as a macroscopic manifestation of the Brownian motion, as known from the Fokker-Plank equation for a stochastic differential equation with a Brownian motion (a Gaussian process), whereas the fractional Laplacian operator $(-\Delta)^\alpha$ is associated with a 2α -stable Lévy motion (a non-Gaussian process) $L_t^{2\alpha}$, $\alpha \in (0, 1)$, (see [11] for a discussion about this microscopic-macroscopic relation). From the observations and experiments related to Lévy flights ([3, 17, 19, 21]), the fractional Laplacian described that a particle could have infinite jumps in an arbitrary time interval with intensity proportional to $\frac{1}{|x-y|^{N+2\alpha}}$, but if the particle jumping is forced to restrict only from one point $x \in \Omega$, a bounded open domain Ω in \mathbb{R}^N , to another point $y \in \Omega$ with the same intensity, then the related process is called the censored stable process and its generator is the regional fractional Laplacian defined in Ω , see the references [5, 6, 16]. In particular, the authors in [3] pointed out that the censored 2α -stable process is conservative and will never approach $\partial\Omega$ when $\alpha \in (0, \frac{1}{2}]$ and for $\alpha \in (\frac{1}{2}, 1)$ that process could approach to the boundary $\partial\Omega$. This indicates that the Dirichlet problem involving the regional fractional Laplacian is well defined for $\alpha \in (\frac{1}{2}, 1)$ and in this note, we pay our attentions on the solutions to related Dirichlet elliptic problem with $\alpha \in (\frac{1}{2}, 1)$.

Throughout this paper, we assume that $\alpha \in (\frac{1}{2}, 1)$, $\beta = 2\alpha - 1$, Ω is a bounded open domain in \mathbb{R}^N ($N \geq 2$) with C^2 boundary $\partial\Omega$ and $\rho(x) = \text{dist}(x, \partial\Omega)$. Denote by $(-\Delta)_\Omega^\alpha$ the regional fractional Laplacian

$$(-\Delta)_\Omega^\alpha u(x) = \lim_{\varepsilon \rightarrow 0^+} (-\Delta)_{\Omega, \varepsilon}^\alpha u(x)$$

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with

$$(-\Delta)_{\Omega,\varepsilon}^\alpha u(x) = -c_{N,\alpha} \int_{\Omega \setminus B_\varepsilon(x)} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} dz,$$

where $c_{N,\alpha} > 0$ coincides the normalized constant of the fractional Laplacian. The main objective of this note is to study the weak solution of elliptic problem

$$\begin{cases} (-\Delta)_\Omega^\alpha u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $f : \Omega \rightarrow \mathbb{R}$. We will concentrate on the existence and uniqueness of the solution of (1.1) in a suitable weak sense when $f \in L^2(\Omega)$ or f belongs to Radon measure space.

When $f \in L^2(\Omega)$, it involves the Hilbert space $H_0^\alpha(\Omega)$ with the scalar product

$$\langle u, v \rangle_\alpha = \frac{c_{N,\alpha}}{2} \int_\Omega \int_\Omega \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{N+2\alpha}} dx dy + \int_\Omega uv dx, \quad \forall u, v \in H_0^\alpha(\Omega).$$

which is the closure of $C_c^2(\bar{\Omega})$ under the norm

$$\|u\|_{H^\alpha(\Omega)} = \sqrt{\langle u, u \rangle_\alpha},$$

which, shown in [10], is equivalent to the Gagliardo norm $\|u\|_{H^\alpha(\Omega)}$ in $H_0^\alpha(\Omega)$

$$\|u\|_{H_0^\alpha(\Omega)} := \left(\frac{c_{N,\alpha}}{2} \int_\Omega \int_\Omega \frac{[u(x) - u(y)]^2}{|x - y|^{N+2\alpha}} dx dy \right)^{\frac{1}{2}}$$

and its scalar product of $\|\cdot\|_{H_0^\alpha(\Omega)}$ is

$$\langle u, v \rangle_{H_0^\alpha(\Omega)} = \frac{c_{N,\alpha}}{2} \int_\Omega \int_\Omega \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{N+2\alpha}} dx dy, \quad \forall u, v \in H_0^\alpha(\Omega).$$

Definition 1.1. (i) When $f \in L^2(\Omega)$, a function $u \in H_0^\alpha(\Omega)$, is a weak solution of (1.1), if

$$\langle u, v \rangle_{H_0^\alpha(\Omega)} = \int_\Omega f(x)v(x) dx, \quad \forall v \in H_0^\alpha(\Omega).$$

(ii) Denote by \mathbb{X}_α the space of functions ξ , continuous up to the boundary, taking zero value on $\partial\Omega$ and verifying

$$\|(-\Delta)_\Omega^\alpha \xi\|_{L^\infty(\Omega)} < \infty,$$

and by $\mathcal{M}(\Omega, \rho^\beta)$ the space of all the Radon measure ν satisfying

$$\int_\Omega \rho^\beta d|\nu| < +\infty.$$

When $f \in \mathcal{M}(\Omega, \rho^\beta)$, a function $u \in L^1(\Omega)$ is a very weak solution of (1.1), if

$$\int_\Omega u(-\Delta)_\Omega^\alpha \xi dx = \int_\Omega \xi df, \quad \forall \xi \in \mathbb{X}_\alpha(\Omega).$$

We notice that $\beta = 1$ if $\alpha = 1$, and in this case the test functions' space corresponding to very weak solution is $C_0^{1,1}(\Omega)$, in which the function could be controlled by the distance function ρ . In the regional fractional case, the test functions space $\mathbb{X}_\alpha(\Omega)$ plays the same role and the function in $\mathbb{X}_\alpha(\Omega)$ has the decay ρ^β , see Lemma 2.5 below.

Now we are ready to state our main theorem on the existence and uniqueness of weak solution for problem (1.1).

Theorem 1.1. (i) Let $f \in L^2(\Omega)$, then problem (1.1) has a unique weak solution u_f such that

$$\|u_f\|_{H_0^\alpha(\Omega)} \leq c_1 \|f\|_{L^2(\Omega)}, \quad (1.2)$$

where $c_1 > 0$.

(ii) Let $f \in \mathcal{M}(\Omega, \rho^\beta)$, then problem (1.1) has a unique very weak solution u_f such that

$$\|u_f\|_{L^1(\Omega)} \leq c_2 \|f\|_{\mathcal{M}(\Omega, \rho^\beta)}, \quad (1.3)$$

where $c_2 > 0$.

For $f \in L^2(\Omega)$ or $f \in \mathcal{M}(\Omega, \rho^\beta)$, a sequence of functions $\{f_n\}_n$ in $C^2(\Omega) \cap C(\bar{\Omega})$ could be chosen to converge to f in $L^2(\Omega)$ and we prove the solution of (1.1) is approached by the classical solution of

$$\begin{cases} (-\Delta)_\Omega^\alpha u = f_n & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this approaching process, the most important tool is the Integral by Parts formula,

$$\begin{aligned} \int_\Omega u(-\Delta)_\Omega^\alpha v \, dx &= \frac{c_{N,\alpha}}{2} \int_\Omega \int_\Omega \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{N+2\alpha}} \, dx dy \\ &= \int_\Omega v(-\Delta)_\Omega^\alpha u \, dx, \quad \forall u, v \in \mathbb{X}_\alpha(\Omega). \end{aligned} \quad (1.4)$$

Thanks to a fractional Hardy-Sobolev inequality from [12], we also show the equivalence between the norms $\|u\|_{H^\alpha(\Omega)}$ and $\|\cdot\|_{H_0^\alpha(\Omega)}$ for functions in $C_0^\infty(\Omega)$.

It is known that $L^1(\Omega, \rho^\beta \, dx)$ is a proper subset of $\mathcal{M}(\Omega, \rho^\beta)$ and we abuse the notation without confusion that $df(x) = f(x)dx$ when $f \in L^1(\Omega, \rho^\beta \, dx)$ in the definition of very weak solution. But the proofs of the existence of very weak solutions to (1.1) are very different for f in $L^1(\Omega, \rho^\beta \, dx)$ and in $\mathcal{M}(\Omega, \rho^\beta)$. For $f \in L^1(\Omega, \rho^\beta \, dx)$, the very weak solution is approached directly by a Cauchy sequence in $L^1(\Omega)$, while in the case of $f \in \mathcal{M}(\Omega, \rho^\beta)$, we have to prove the approximations is uniformly bounded in $L^1(\Omega)$ and uniformly integrable, then Dunford-Pettis Theorem is applied to derive the very weak solution of (1.1). The elliptic problems involving measure data with second order operators have been extensively studied in [1, 2, 14, 20, 23] and the reference therein, and recently, the elliptic problems involving the fractional Laplacian have been investigated by [7, 8, 9].

Finally, we make use of the nonlocal characteristic property to build an Integral by Parts Formula for the solution u of

$$\begin{cases} (-\Delta)_\Omega^\alpha u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases} \quad (1.5)$$

for $f \in C^2(\Omega) \cap C(\bar{\Omega})$ and $g \in C^2(\partial\Omega)$. That is,

$$\int_\Omega u(-\Delta)_\Omega^\alpha \xi \, dx = \int_\Omega f(x)\xi(x) \, dx + \int_{\partial\Omega} \frac{\partial^\beta \xi(x)}{\partial \vec{n}_x^\beta} g(x) \, d\omega(x), \quad \forall \xi \in \mathbb{X}_\alpha(\Omega) \cap \mathbb{D}_\beta, \quad (1.6)$$

where \vec{n}_x is the unit exterior normal vector of Ω at point $x \in \partial\Omega$ and

$$\mathbb{D}_\beta = \bigcup_{\tau \geq \beta} \{\phi_1 \rho^\tau + \phi_2 : \phi_1, \phi_2 \in C^2(\bar{\Omega})\}. \quad (1.7)$$

Here we note that

$$\frac{\partial^\beta \xi(x)}{\partial \vec{n}_x^\beta} = \lim_{t \rightarrow 0^+} \frac{\xi(x) - \xi(x - t\vec{n}_x)}{t^\beta} = - \lim_{t \rightarrow 0^+} \frac{\xi(x - t\vec{n}_x)}{t^\beta}$$

and an Integral by Parts Formula provided in [15] states as follows

$$\int_{\Omega} w(-\Delta)_{\Omega}^{\alpha} v \, dx = \int_{\Omega} v(-\Delta)_{\Omega}^{\alpha} w \, dx, \quad \forall w, v \in \mathbb{D}_{\beta}. \quad (1.8)$$

The paper is organized as follows. In Section §2, we study the solutions of (1.1) with $f \in L^{\infty}$, including the existence and uniqueness of classical solution, the boundary regularities and also provides some important estimates for proving (1.4). Section §3 is devoted to give an Integration by Parts Formula for $u, v \in \mathbb{X}_{\alpha}(\Omega)$, then we obtain the existence and uniqueness of weak solution of (1.1) with zero boundary data and $f \in L^2(\Omega)$. In Section §4, we prove the very weak solution of (1.1) for $f \in L^1(\Omega, \rho^{\beta} \, dx)$ and $f \in \mathcal{M}(\Omega, \rho^{\beta})$. Finally, we provide Integral by Parts Formula (1.6) for the solution of (1.1) with general boundary data in Section §5.

2. PRELIMINARY

The purpose of this section is to introduce some preliminaries on the classical solution of (1.1). We start it by the Comparison Principle. In what follows, we denote by c_i a generic positive constant.

Lemma 2.1. *Assume that g is continuous on $\partial\Omega$ and $f_i : \Omega \rightarrow \mathbb{R}$ with $i = 1, 2$ are continuous functions satisfying*

$$f_1 \geq f_2 \quad \text{in } \Omega.$$

Let u_1 and u_2 be the solutions of (1.5) with $f = f_1$ and f_2 , respectively. Then

$$u_1 \geq u_2 \quad \text{in } \Omega. \quad (2.1)$$

Furthermore, if $f \equiv 0$, $g \equiv 0$, then problem (1.5) only has zero solution.

Proof. By contradiction, if (2.1) fails, denoting $w = u_1 - u_2$, there exists $x_0 \in \Omega$ such that

$$w(x_0) = u_1(x_0) - u_2(x_0) = \min_{x \in \Omega} w(x) < 0.$$

Combining with $w = 0$ on $\partial\Omega$, we observe that

$$(-\Delta)_{\Omega}^{\alpha} w(x_0) = \int_{\Omega} \frac{w(x_0) - w(y)}{|x_0 - y|^{N+2\alpha}} \, dy < 0.$$

But

$$(-\Delta)_{\Omega}^{\alpha} w(x_0) = f_1(x_0) - f_2(x_0) \geq 0,$$

which is impossible. \square

Our main aim here is give the regularity up to the boundary of the solution of (1.1).

Theorem 2.1. *Assume that $\alpha \in (\frac{1}{2}, 1)$, $f \in L^{\infty}(\Omega)$, $f_{\pm} = \max\{\pm f, 0\}$ and $\rho(x) = \text{dist}(x, \partial\Omega)$. Then problem (1.1) admits a unique solution u_f such that*

$$-c_3 \|f_{-}\|_{L^{\infty}(\Omega)} \rho(x)^{\beta} \leq u_f(x) \leq c_3 \|f_{+}\|_{L^{\infty}(\Omega)} \rho(x)^{\beta} \quad x \in \Omega. \quad (2.2)$$

Moreover,

(i) for $\theta \in (0, 2\alpha)$ and an open set \mathcal{O} satisfying $d_{\mathcal{O}} := \text{dist}(\mathcal{O}, \partial\Omega) > 0$, there exists $c_4 > 0$ dependent of $d_{\mathcal{O}}$ and θ such that

$$\|u_f\|_{C^{\theta}(\mathcal{O})} \leq c_4 \|f\|_{L^{\infty}(\Omega)}; \quad (2.3)$$

(ii) there exists $c_5 > 0$ independent of f such that

$$\|u_f\|_{C^{\beta}(\bar{\Omega})} \leq c_5 \|f\|_{L^{\infty}(\Omega)}. \quad (2.4)$$

In order to consider (1.1), we need the following uniformly estimates. Denote by $G_{\Omega,\alpha}$ the Green kernel of $(-\Delta)_\Omega^\alpha$ in $\Omega \times \Omega$ and by $\mathbb{G}_{\Omega,\alpha}[\cdot]$ the Green operator defined as

$$\mathbb{G}_{\Omega,\alpha}[f](x) = \int_{\Omega} G_{\Omega,\alpha}(x, y) f(y) dy.$$

Lemma 2.2. *Let $\alpha \in (\frac{1}{2}, 1)$ and $f \in L^\infty(\Omega)$, then $\mathbb{G}_{\Omega,\alpha}[f]$ is the unique solution of problem (1.1) and*

$$|\mathbb{G}_{\Omega,\alpha}[f](x)| \leq c_6 \rho(x)^\beta, \quad \forall x \in \Omega. \quad (2.5)$$

Proof. The uniqueness follows by Lemma 2.1. We observe that $\mathbb{G}_{\Omega,\alpha}[f]$ is a solution of $(-\Delta)_\Omega^\alpha w = f$ in Ω and for any $(x, y) \in \Omega \times \Omega$ with $x \neq y$,

$$G_{\Omega,\alpha}(x, y) \leq c_7 \min \left\{ \frac{1}{|x - y|^{N-2\alpha}}, \frac{\rho(x)^\beta \rho(y)^\beta}{|x - y|^{N-1+\beta}} \right\}, \quad (2.6)$$

see [5]. Then we have that

$$\begin{aligned} |\mathbb{G}_{\Omega,\alpha}[f](x)| &\leq c_7 \int_{\Omega} \frac{\rho(x)^\beta \rho(y)^\beta}{|x - y|^{N-1+\beta}} |f(y)| dy \\ &\leq c_7 \rho(x)^\beta \|f\|_{L^\infty(\Omega)} \int_{\Omega} \frac{\rho(y)^\beta}{|x - y|^{N-1+\beta}} dy \\ &\leq c_8 \|f\|_{L^\infty(\Omega)} \rho(x)^\beta, \quad \forall x \in \Omega. \end{aligned}$$

Hence $\mathbb{G}_{\Omega,\alpha}[f]$ is a solution of (1.1) verifying (2.5). \square

In what follows, we denote

$$u_f = \mathbb{G}_{\Omega,\alpha}[f].$$

Lemma 2.3. *For any $x_0 \in \Omega$ and $\theta \in (0, 2\alpha)$, there exists $c_9 > 0$ independent of $\rho(x_0)$ such that*

$$\|u_f\|_{C^\theta(B_{\rho_0}(x_0))} \leq c_9 \rho_0^{\beta-\theta} \|f\|_{L^\infty(\Omega)}, \quad (2.7)$$

where $\rho_0 = \rho(x_0)/3$.

Proof. For $x_0 \in \Omega$, we denote that $\Omega_0 = \{y \in \mathbb{R}^N : x_0 + \rho_0 y \in \Omega\}$ and

$$v_f(x) = u_f(x_0 + \rho_0 x), \quad \forall x \in \mathbb{R}^N,$$

then by Lemma 2.2, we have that

$$\|v_f\|_{L^\infty(B_2(0))} = \|u_f\|_{L^\infty(B_{2\rho_0}(x_0))} \leq c_6 \|f\|_{L^\infty(\Omega)} \rho_0^\beta$$

and for $x \in B_2(0)$,

$$\begin{aligned} (-\Delta)_{\Omega_0}^\alpha v_f(x) &= \text{P.V.} c_{N,\alpha} \int_{\Omega_0} \frac{u_f(x_0 + \rho_0 x) - u_f(x_0 + \rho_0 y)}{|x - y|^{N+2\alpha}} dy \\ &= \rho_0^{2\alpha} (-\Delta)_\Omega^\alpha u_f(x_0 + \rho_0 x) = \rho_0^{2\alpha} f(x_0 + \rho_0 x) \end{aligned}$$

and

$$\begin{aligned} (-\Delta)^\alpha v_f(x) &= (-\Delta)_{\Omega_0}^\alpha v_f(x) + v_f(x) \phi_0(x) \\ &= \rho_0^{2\alpha} f(x_0 + \rho_0 x) + v_f(x) \phi_0(x), \quad \forall x \in B_2(0), \end{aligned}$$

where $\phi_0(x) = c_{N,\alpha} \int_{\mathbb{R}^N \setminus \Omega_0} \frac{1}{|x - y|^{N+2\alpha}} dy$. Since $B_3(0) \subset \Omega_0$, we have that

$$\phi_0(x) \leq c_{N,\alpha}, \quad \forall x \in B_2(0).$$

Then by [22, Proposition 2.3], we have that

$$\begin{aligned}\|v_f\|_{C^\gamma(B_1(0))} &\leq c_{10} (\|\rho_0^{2\alpha} f(x_0 + \rho_0 \cdot) + v_f \phi_0\|_{L^\infty(B_2(0))} + \|v_f\|_{L^\infty(B_2(0))}) \\ &\leq c_{10} (\rho_0^{2\alpha} \|f\|_{L^\infty(\Omega)} + \|v_f\|_{L^\infty(B_2(0))}).\end{aligned}$$

Since

$$\|v_f\|_{L^\infty(B_2(0))} = \|u_f\|_{L^\infty(B_{2\rho_0}(x_0))} \leq c_{11} \rho_0^\beta \|f\|_{L^\infty(\Omega)},$$

then we have that

$$\|u_f\|_{C^\theta(B_{\rho_0}(x_0))} \leq c_{12} \rho_0^{\beta-\theta} \|f\|_{L^\infty(\Omega)}.$$

The proof ends. \square

Lemma 2.4. *Let*

$$\phi(x) = \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|x - y|^{N+2\alpha}} dy, \quad (2.8)$$

then $\phi \in C_{\text{loc}}^{0,1}(\Omega)$ and for some $c_{13} > 1$

$$\frac{1}{c_{13}} \rho(x)^{-2\alpha} \leq \phi(x) \leq c_{13} \rho(x)^{-2\alpha}, \quad \forall x \in \Omega. \quad (2.9)$$

Proof. For $x_1, x_2 \in \Omega$ and any $z \in \mathbb{R}^N \setminus \Omega$, we have that

$$|z - x_1| \geq \rho(x_1) + \rho(z), \quad |z - x_2| \geq \rho(x_2) + \rho(z)$$

and

$$||z - x_1|^{N+2\alpha} - |z - x_2|^{N+2\alpha}| \leq c_{14} |x_1 - x_2| (|z - x_1|^{N+2\alpha-1} + |z - x_2|^{N+2\alpha-1}),$$

for some $c_{14} > 0$ independent of x_1 and x_2 . Then

$$\begin{aligned}|\phi(x_1) - \phi(x_2)| &\leq \int_{\mathbb{R}^N \setminus \Omega} \frac{||z - x_2|^{N+2\alpha} - |z - x_1|^{N+2\alpha}|}{|z - x_1|^{N+2\alpha} |z - x_2|^{N+2\alpha}} dz \\ &\leq c_{14} |x_1 - x_2| \left[\int_{\mathbb{R}^N \setminus \Omega} \frac{dz}{|z - x_1| |z - x_2|^{N+2\alpha}} + \int_{\mathbb{R}^N \setminus \Omega} \frac{dz}{|z - x_1|^{N+2\alpha} |z - x_2|} \right].\end{aligned}$$

By direct computation, we have that

$$\begin{aligned}\int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|z - x_1| |z - x_2|^{N+2\alpha}} dz &\leq \int_{\mathbb{R}^N \setminus B_{\rho(x_1)}(x_1)} \frac{1}{|z - x_1|^{N+2\alpha+1}} dz \\ &\quad + \int_{\mathbb{R}^N \setminus B_{\rho(x_2)}(x_2)} \frac{1}{|z - x_2|^{N+2\alpha+1}} dz \\ &\leq c_{15} [\rho(x_1)^{-1-2\alpha} + \rho(x_2)^{-1-2\alpha}]\end{aligned}$$

and similar to obtain that

$$\int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|z - x_1|^{N+2\alpha} |z - x_2|} dz \leq c_{15} [\rho(x_1)^{-1-2\alpha} + \rho(x_2)^{-1-2\alpha}].$$

Then

$$|\phi(x_1) - \phi(x_2)| \leq c_{14} c_{15} [\rho(x_1)^{-1-2\alpha} + \rho(x_2)^{-1-2\alpha}] |x_1 - x_2|,$$

that is, ϕ is $C^{0,1}$ locally in Ω .

We next prove (2.9). Without loss of generality, we may assume that $0 \in \partial\Omega$, the inside pointing normal vector at 0 is $e_N = (0, \dots, 0, 1) \in \mathbb{R}^N$ and let $s \in (0, \frac{1}{4})$ such that $\mathbb{R}^N \setminus \Omega \subset \mathbb{R}^N \setminus B_s(se_N)$ and for $t > 0$, we denote the cone

$$A_t = \{y = (y', y_N) \in \mathbb{R}^N : y_N \leq s - t|y'|\}.$$

We observe that there is $c_0 > 0$ such that

$$[A_{t_0} \cap (B_1(se_N) \setminus B_{2s}(se_N))] \subset \mathbb{R}^N \setminus \Omega.$$

By the definition of ϕ , we have that

$$\phi(se_N) = \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|se_N - y|^{N+2\alpha}} dy \leq \int_{\mathbb{R}^N \setminus B_s(se_N)} \frac{1}{|se_N - y|^{N+2\alpha}} dy \leq c_{16} s^{-2\alpha}.$$

On the other hand, we have that

$$\int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|se_N - y|^{N+2\alpha}} dy \geq \int_{A_{c_0} \cap (B_1(se_N) \setminus B_{2s}(se_N))} \frac{1}{|se_N - y|^{N+2\alpha}} dy \geq c_{17} s^{-2\alpha}.$$

The proof ends. \square

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Lemma 2.2, u_f is the unique solution of (1.1). Since $\mathbb{G}_{\Omega, \alpha}[f_+]$, $-\mathbb{G}_{\Omega, \alpha}[f_-]$ are solutions of (1.1) replaced f by f_+ and f_- respectively. Then (2.2) follows by Lemma 2.1.

Proof of (2.3). Let $\tilde{w} = w$ in Ω , $\tilde{w} = 0$ in $\mathbb{R}^N \setminus \bar{\Omega}$, we observe that

$$(-\Delta)^\alpha \tilde{w}(x) = (-\Delta)_\Omega^\alpha w(x) + w(x)\phi(x), \quad \forall x \in \Omega,$$

where ϕ is defined by (2.8). It follows by Lemma 2.4, $\phi \in C_{\text{loc}}^{0,1}(\Omega)$. Then

$$(-\Delta)^\alpha \tilde{w}(x) = f(x) + w(x)\phi(x), \quad \forall x \in \Omega.$$

Let \mathcal{O}_1 be a C^2 open set such that

$$\mathcal{O} \subset \mathcal{O}_1, \quad \text{dist}(\mathcal{O}_1, \partial\Omega) = d_{\mathcal{O}}/2, \quad \text{and} \quad \text{dist}(\mathcal{O}, \partial\mathcal{O}_1) = d_{\mathcal{O}}/2.$$

By [9, Lemma 3.1], for any $\theta \in (0, 2\alpha)$, we have that

$$\|\tilde{w}\|_{C^\theta(\mathcal{O})} \leq c_{18} [\|\tilde{w}\|_{L^\infty(\mathcal{O}_1)} + \|\tilde{w}\|_{L^1(\Omega)} + \|f + \tilde{w}\phi\|_{L^\infty(\mathcal{O}_1)}]. \quad (2.10)$$

By Lemma 2.2 and Lemma 2.4, we obtain that

$$\|\tilde{w}\|_{L^\infty(\mathcal{O}_1)} + \|\tilde{w}\|_{L^1(\Omega)} \leq \|f\|_{L^\infty(\Omega)}$$

and

$$|\tilde{w}(x)|\phi(x) \leq c_{19}\rho(x)^{2\alpha-1}\|f\|_{L^\infty(\Omega)}\rho(x)^{-2\alpha} \leq c_{19}\rho(x)^{-1}\|f\|_{L^\infty(\Omega)},$$

then

$$\|f + \tilde{w}\phi\|_{L^\infty(\mathcal{O}_1)} \leq \|f\|_{L^\infty(\Omega)} + \|\tilde{w}\phi\|_{L^\infty(\mathcal{O}_1)} \leq c_{20}d_{\mathcal{O}}^{-1}\|f\|_{L^\infty(\Omega)}.$$

Then (2.3) holds.

Proof of (2.4). Taking $\theta = 2\alpha - 1$ in Lemma 2.3, we have that

$$\frac{u(x) - u(y)}{|x - y|^\theta} \leq c_{21}\|f\|_{L^\infty(\Omega)} \quad (2.11)$$

for all x, y such that $y \in B_R(x)$ with $R = \rho(x)/3$. We next show that (2.11) holds for all $x, y \in \bar{\Omega}$ with some renewed constant.

Indeed, we observe that after a Lipschitz change of coordinates, the bound (2.11) remains the same except for the value of the constant c . Then we can flatten the boundary near $x_0 \in \partial\Omega$ to assume that $\Omega \cap B_{\rho_0}(x_0) = \{x_n > 0\} \cap B_1(0)$. Thus, (2.11) holds for all x, y satisfying $|x - y| \leq \gamma x_n$ for some $\gamma = \gamma(\Omega) \in (0, 1)$ dependent of the Lipschitz mapping.

Let $z = (z', z_n)$ and $w = (w', w_n)$ be two points in $\{x_n > 0\} \cap B_{1/4}(0)$ and $r = |z - w|$. Denote that $\bar{z} = (z', z_n + r)$, $\bar{w} = (w', w_n + r)$ and $z_k = (1 - \gamma^k)z + \gamma^k \bar{z}$ and $w_k =$

$\gamma^k w + (1 - \gamma^k) \bar{w}$, $k \geq 0$. Then, using the bound (2.11) whenever $|x - y| \leq \gamma x_n$, we have that

$$|u(z_{k+1}) - u(z_k)| \leq c_{22} |z_{k+1} - z_k|^\theta = c_{21} |\gamma^k (z - \bar{z})(\gamma - 1)|^\theta \leq c_{22} \gamma^k |z - \bar{z}|.$$

Moreover, since $x_n > r$ in all the segment joining \bar{z} and \bar{w} , splitting this segment into a bounded number of segments of length less than γr , we obtain that

$$|u(\bar{z}) - u(\bar{w})| \leq c_{23} |\bar{z} - \bar{w}|^\theta \leq c_{23} r^\theta.$$

Therefore,

$$\begin{aligned} |u(z) - u(w)| &\leq \sum_{k \geq 0} |u(z_{k+1}) - u(z_k)| + |u(\bar{z}) - u(\bar{w})| + \sum_{k \geq 0} |u(w_{k+1}) - u(w_k)| \\ &\leq (c_{24} \sum_{k \geq 0} (\gamma^k r)^\theta + c_{25} r^\theta) (\|u\|_{L^\infty(\mathbb{R}^N)} + \|g\|_{L^\infty(\Omega)}) \\ &\leq c_{26} (\|u\|_{L^\infty(\mathbb{R}^N)} + \|g\|_{L^\infty(\Omega)}) |z - w|^\theta, \end{aligned}$$

which ends the proof. \square

For a unbounded nonhomogeneous term f , we have that

Lemma 2.5. *Assume that f is a $C_{loc}^\gamma(\Omega)$ function satisfying*

$$|f(x)| \leq c_{27} \rho(x)^{-\beta}, \quad \forall x \in \Omega,$$

where $\gamma \in (0, 1)$. Then problem (1.1) has a unique solution u_f satisfying

$$|u_f(x)| \leq c_{28} \|f \rho^\beta\|_{L^\infty(\Omega)} \rho^\beta(x), \quad \forall x \in \Omega. \quad (2.12)$$

Proof. The uniqueness follows by Lemma 2.1. It is known that $\mathbb{G}_{\Omega, \alpha}[f]$ is a solution of $(-\Delta)_\Omega^\alpha w = f$ in Ω . From (2.6), we have that for $x \in \Omega$,

$$\begin{aligned} |\mathbb{G}_{\Omega, \alpha}[f](x)| &\leq c_7 \int_\Omega \frac{\rho(x)^\beta \rho(y)^\beta}{|x - y|^{N-2+2\alpha}} |f(y)| dy \\ &\leq c_7 \rho(x)^\beta \|f \rho^\beta\|_{L^\infty(\Omega)} \int_\Omega \frac{1}{|x - y|^{N-2+2\alpha}} dy \\ &\leq c_7 \|f \rho^\beta\|_{L^\infty(\Omega)} \rho(x)^\beta \int_{B_{d_0}(x)} \frac{1}{|x - y|^{N-2+2\alpha}} dy, \end{aligned}$$

where $d_0 = \sup_{x, y \in \Omega} |x - y|$ and $\int_{B_{d_0}(x)} \frac{1}{|x - y|^{N-2+2\alpha}} dy < +\infty$ by the fact that $N - 2 + 2\alpha < N$. Therefore, we obtain that $\mathbb{G}_{\Omega, \alpha}[f]$ is a solution of (1.1) satisfying (2.12). \square

Remark 2.1. We remark that (2.12) holds for $v \in \mathbb{X}_\alpha(\Omega)$. In fact, let $f = (-\Delta)_\Omega^\alpha v$, which satisfies

$$\|f \rho^\beta\|_{L^\infty(\Omega)} < +\infty.$$

The next proposition plays an important role in the proof of Integration by Parts Formula with nonzero Dirichlet boundary condition. For this purpose, we introduce some notations. Denote

$$\Omega_\delta := \{x \in \Omega : \rho(x) > \delta\} \quad \text{and} \quad A_\delta := \{x \in \Omega : \rho(x) < \delta\}. \quad (2.13)$$

Since Ω is C^2 , there exists $\delta_0 > 0$ such that Ω_δ is C^2 for any $\delta \in (0, \delta_0]$ and it is known that for any $x \in \partial\Omega_\delta$, there exists $x^* \in \partial\Omega$ such that

$$|x - x^*| = \rho(x) \quad \text{and} \quad x = x^* + \rho(x) \vec{n}_{x^*}.$$

Proposition 2.1. *Assume that $f \in C^2(\Omega) \cap C(\bar{\Omega})$ and $g \in C^2(\partial\Omega)$. Let u be the classical solution of (1.5). Then $u \in C^2(\Omega) \cap C^\beta(\bar{\Omega})$. Furthermore, for $\delta \in (0, \delta_0)$, there exists $c_{29} > 0$ such that*

$$|u(x) - u(y)| < c_{29}\rho(x)^{\beta-1}|x - y|, \quad \forall x \in \Omega_\delta, \forall y \in A_\delta. \quad (2.14)$$

Proof. *To prove $u \in C^2(\Omega) \cap C^\beta(\bar{\Omega})$. Here we only have to prove $u \in C^2(\Omega) \cap C^\beta(\bar{\Omega})$ in the case that $g \equiv 0$. In fact, since Ω is C^2 and $g \in C^2(\partial\Omega)$, then there exists $G \in C^2(\bar{\Omega})$ such that*

$$G = g \quad \text{on } \partial\Omega.$$

Now we only consider the regularity of $u - G$, which is the solution of

$$\begin{aligned} (-\Delta)_\Omega^\alpha u &= f - (-\Delta)_\Omega^\alpha G & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (2.15)$$

So it follows by Theorem 2.1 that $u \in C^\beta(\bar{\Omega})$.

We next prove $u \in C^2(\Omega)$. Extend the function u by zero in $\mathbb{R}^N \setminus \Omega$, still denote it u , and then

$$(-\Delta)^\alpha u(x) = (-\Delta)_\Omega^\alpha u(x) + u(x)\phi(x) = f(x) + u(x)\phi(x), \quad \forall x \in \Omega,$$

where $(-\Delta)^\alpha$ is the global fractional Laplacian. From Lemma 2.4, $\phi \in C_{loc}^{0,1}(\Omega)$, applying [22, Corollary 1.6] with $\theta < 1 + 2\alpha$, we have that

$$u \in C_{loc}^\theta(\Omega).$$

This means $u \in C^2(\Omega)$ since we can choose $\theta > 2$.

To prove (2.14). By the compactness of $\partial\Omega$, we only consider a point $x_0 \in \partial\Omega$ and for simplicity, we can assume that $x_0 = 0$. Let $x = t\vec{n}_0$ with $t \in (0, \delta)$ and $y \in B_{\frac{\delta}{3}}(\delta\vec{n}_0) \cap \Omega_\delta$, for any $t \in (0, \delta]$, there exists n depending on t such that

$$t \in \left(\frac{\delta}{3^{n+1}}, \frac{\delta}{3^n} \right)$$

and then we choose $x_k = x + \frac{1}{3^k}(y - x)$, $k = 0, 1, \dots, n$. We observe that

$$\rho(x_k) \geq \frac{\delta}{3^k}, \quad k = 0, 1, \dots, n.$$

From Lemma 2.3, it follows that for $k = 0, 1, \dots, n$

$$\|w\|_{C^{0,1}(B_{\rho(x_k)/2}(x_k))} \leq c_{30}\rho(x_k)^{\beta-1}\|f\|_{L^\infty(\Omega)} \leq c_{30}\left(\frac{\delta}{3^k}\right)^{\beta-1}\|f\|_{L^\infty(\Omega)}. \quad (2.16)$$

It is obvious that $x_0 = y$ and

$$|x_k - x_{k+1}| < \frac{|x - y|}{3^k}.$$

Then we have that for $k = 0, 1, \dots, n$,

$$\begin{aligned} |w(x_k) - w(x_{k-1})| &\leq \|w\|_{C^{0,1}(B_{\frac{\rho(x_k)}{2}}(x_k))} |x_k - x_{k-1}| \\ &\leq c_{30}\left(\frac{\delta}{3^k}\right)^{\beta-1} |x_k - x_{k-1}| \\ &\leq c_{30}t^{\beta-1} |x_k - x_{k-1}|, \end{aligned}$$

therefore,

$$|w(x) - w(x_0)| \leq c_7 t^{(1-\beta)} \sum_{k=0}^n \frac{1}{3^k} |x - y| \leq c_{31} t^{\beta-1} |x - y|,$$

where $c_{31} > 0$ is independent of t . So for some $c_{32} > 0$, we have that

$$|w(x) - w(y)| \leq c_{32}\rho(x)^{\beta-1}|x - y|. \quad (2.17)$$

For $y \in \Omega_\delta \setminus B_{\frac{\delta}{3}}(\delta\vec{n}_0)$, we may choose $y' \in B_{\frac{\delta}{3}}(\delta\vec{n}_0) \cap \Omega_\delta$. There are at most N_0 points $y_k \in \Omega_\delta$ connecting y and y' such that

$$\frac{\delta}{3} \leq |y_k - y_{k-1}| \leq \frac{\delta}{2}.$$

We see that

$$|w(y) - w(y')| \leq c_{33}\delta^{\beta-1}|y - y'|.$$

From (2.17), we see that

$$|w(x) - w(y')| \leq c_{32}\rho(x)^{\beta-1}|x - y'|.$$

Since $|y - y'| \geq \frac{\delta}{3}$ and $|x - y| > \delta$, then

$$\begin{aligned} |w(x) - w(y)| &\leq |w(x) - w(y')| + |w(y) - w(y')| \\ &\leq c_{33}\delta^{\beta-1}|y - y'| + c_{32}\rho(x)^{\beta-1}|x - y'| \\ &\leq c_{34}\rho(x)^{\beta-1}|x - y|. \end{aligned}$$

We finish the proof. \square

Lemma 2.6. *Assume that $f \in C^2(\Omega) \cap C(\bar{\Omega})$, $g \in C^2(\partial\Omega)$ and w is the classical solution of (1.5). Then*

$$\int_{\Omega} \int_{\Omega} \frac{[u(x) - u(y)]^2}{|x - y|^{N+2\alpha}} dx dy < +\infty. \quad (2.18)$$

Proof. From the interior regularity, we know that $u \in C^2(\Omega) \cap C^\beta(\bar{\Omega})$. From [16, Theorem 3.4], it infers that

$$\begin{aligned} &\frac{c_{N,\alpha}}{2} \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{[u(x) - u(y)]^2}{|x - y|^{N+2\alpha}} dx dy \\ &= \int_{\Omega_\delta} u(x)(-\Delta)_{\Omega_\delta}^\alpha u(x) dx \\ &= c_{N,\alpha} \int_{\Omega_\delta} \int_{A_\delta} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} u(x) dy dx + \int_{\Omega_\delta} u(x)(-\Delta)_\Omega^\alpha u(x) dx \\ &= c_{N,\alpha} \int_{\Omega_\delta} \int_{A_\delta} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} u(x) dy dx + \int_{\Omega_\delta} u(x)f(x) dx. \end{aligned} \quad (2.19)$$

We observe that

$$\left| \int_{\Omega_\delta} u(x)f(x) dx \right| \leq |\Omega| \|u\|_{L^\infty(\Omega)} \|f\|_{L^\infty(\Omega)}. \quad (2.20)$$

From Proposition 2.1, we derive that

$$\begin{aligned} \int_{\Omega_\delta} \int_{A_\delta} \frac{|u(x) - u(y)|}{|x - y|^{N+2\alpha}} |u(x)| dy dx &\leq \|u\|_{L^\infty(\Omega)} \int_{A_\delta} \int_{\Omega_\delta} \frac{|u(x) - u(y)|}{|x - y|^{N+2\alpha}} dx dy \\ &\leq c_{35} \|u\|_{L^\infty(\Omega)} \int_{A_\delta} \rho(y)^{\beta-1} \int_{\Omega_\delta} \frac{1}{|x - y|^{N+2\alpha-1}} dx dy \\ &\leq c_{36} \|u\|_{L^\infty(\Omega)} \int_{A_\delta} \rho(y)^{\beta-1} \int_{\delta-\rho(y)}^{d_0} \frac{1}{r^{2\alpha}} dr dy \\ &\leq c_{37} \|u\|_{L^\infty(\Omega)} \int_{A_\delta} \rho(y)^{\beta-1} (\delta - \rho(y))^{-\beta} dy. \end{aligned}$$

Since Ω is C^2 , then for $t \in (0, \delta)$ and $\delta \leq \delta_0$, we have that

$$\frac{1}{2}|\partial\Omega| \leq |\partial\Omega_t| \leq 2|\partial\Omega|$$

and by Fubini's theorem

$$\begin{aligned} \int_{A_\delta} \rho(y)^{\beta-1} (\delta - \rho(y))^{-\beta} dy &= \int_0^\delta t^{\beta-1} (\delta - t)^{-\beta} |\partial\Omega_t| dt \\ &\leq 2|\partial\Omega| \int_0^\delta t^{\beta-1} (\delta - t)^{-\beta} dt \\ &= 2|\partial\Omega| \int_0^1 t^{\beta-1} (1 - t)^{-\beta} dt. \end{aligned}$$

Therefore, for some $c_{38} > 0$ independent of δ there holds that

$$\int_{\Omega_\delta} \int_{A_\delta} \frac{|u(x) - u(y)|}{|x - y|^{N+2\alpha}} |u(x)| dy dx < c_{38}, \quad (2.21)$$

thus, together with (2.19)-(2.20), we derive that

$$\int_{\Omega} \int_{\Omega} \frac{[u(x) - u(y)]^2}{|x - y|^{N+2\alpha}} dx dy < +\infty.$$

The proof ends. \square

Corollary 2.1. *Assume that $f, h \in C^2(\Omega) \cap C(\bar{\Omega})$ and u, w are the classical solution of (1.1) with nonhomogeneous nonlinearities f and h , respectively.*

Then

$$\lim_{\delta \rightarrow 0^+} \int_{\Omega_\delta} \int_{A_\delta} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} w(y) dy dx = 0. \quad (2.22)$$

Proof. From Theorem 2.1,

$$|w(x)| \leq c_{15} \rho(x)^\beta, \quad \forall x \in \Omega.$$

Thus,

$$\begin{aligned} \int_{\Omega_\delta} \int_{A_\delta} \frac{|u(x) - u(y)|}{|x - y|^{N+2\alpha}} |w(y)| dy dx &\leq \|w\|_{L^\infty(A_\delta)} \int_{A_\delta} \int_{\Omega_\delta} \frac{|u(x) - u(y)|}{|x - y|^{N+2\alpha}} dx dy \\ &\leq c_{15} \delta^\beta \int_{A_\delta} \rho^{\beta-1}(y) (\delta - \rho(y))^{-\beta} dy. \end{aligned}$$

By (2.21), we have that

$$\int_{\Omega_\delta} \int_{A_\delta} \frac{|u(x) - u(y)|}{|x - y|^{N+2\alpha}} |w(y)| dy dx \leq c_{38} \delta^\beta,$$

then (2.22) holds. \square

3. ZERO BOUNDARY DATA

3.1. Classical solution. In this subsection, we concentrate on the classical solution of (1.1) when $f \in C^2(\Omega) \cap C(\bar{\Omega})$.

Proposition 3.1. *Assume that $f \in C^2(\Omega) \cap C(\bar{\Omega})$ and u is the classical solution of (1.1). Then*

$$\int_{\Omega} u(-\Delta)_\Omega^\alpha v dx = \int_{\Omega} f(x) v(x) dx, \quad \forall v \in \mathbb{X}_\alpha(\Omega). \quad (3.1)$$

Proof. Let $h(x) = (-\Delta)_\Omega^\alpha v(x)$ and h_n be a sequence of $C^2(\Omega) \cap C(\bar{\Omega})$ functions such that

$$\lim_{n \rightarrow \infty} \|h_n - h\|_{L^\infty(\Omega)} = 0.$$

Let v_n be the solution of

$$\begin{cases} (-\Delta)_\Omega^\alpha u = h_n & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.2)$$

and then

$$\|v_n\|_{C^\beta(\Omega)} \leq c_{16} \|h\|_{L^\infty(\Omega)}.$$

From Lemma 2.6, we have that

$$\int_\Omega \int_\Omega \frac{[u(x) - u(y)]^2}{|x - y|^{N+2\alpha}} dx dy < +\infty \quad \text{and} \quad \int_\Omega \int_\Omega \frac{[v_n(x) - v_n(y)]^2}{|x - y|^{N+2\alpha}} dx dy < +\infty,$$

which imply that

$$\int_{\Omega_\delta} \int_{\Omega_\delta} \frac{[u(x) - u(y)][v_n(x) - v_n(y)]}{|x - y|^{N+2\alpha}} dx dy < +\infty.$$

From [16, Theorem 3.4], it infers that

$$\begin{aligned} & \frac{c_{N,\alpha}}{2} \int_\Omega \int_\Omega \frac{[u(x) - u(y)][v_n(x) - v_n(y)]}{|x - y|^{N+2\alpha}} dx dy \\ &= \frac{c_{N,\alpha}}{2} \lim_{\delta \rightarrow 0^+} \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{[u(x) - u(y)][v_n(x) - v_n(y)]}{|x - y|^{N+2\alpha}} dx dy \\ &= \lim_{\delta \rightarrow 0^+} \int_{\Omega_\delta} v_n(x) (-\Delta)_{\Omega_\delta}^\alpha u(x) dx \\ &= \int_\Omega v_n(x) (-\Delta)_\Omega^\alpha u(x) dx + c_{N,\alpha} \lim_{\delta \rightarrow 0^+} \int_{\Omega_\delta} \int_{A_\delta} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} v_n(x) dy dx \\ &= \int_\Omega v_n(x) f(x) dx + c_{N,\alpha} \lim_{\delta \rightarrow 0^+} \int_{\Omega_\delta} \int_{A_\delta} \frac{[u(x) - u(y)][v_n(x) - v_n(y)]}{|x - y|^{N+2\alpha}} dy dx \\ &\quad + c_{N,\alpha} \lim_{\delta \rightarrow 0^+} \int_{\Omega_\delta} \int_{A_\delta} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} v_n(y) dy dx \end{aligned}$$

and by Corollary 2.1, we have that

$$c_{N,\alpha} \lim_{\delta \rightarrow 0^+} \int_{\Omega_\delta} \int_{A_\delta} \frac{[u(x) - u(y)][v_n(x) - v_n(y)]}{|x - y|^{N+2\alpha}} dy dx = 0$$

and

$$c_{N,\alpha} \lim_{\delta \rightarrow 0^+} \int_{\Omega_\delta} \int_{A_\delta} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} v_n(y) dy dx = 0.$$

Therefore,

$$\frac{c_{N,\alpha}}{2} \int_\Omega \int_\Omega \frac{[u(x) - u(y)][v_n(x) - v_n(y)]}{|x - y|^{N+2\alpha}} dx dy = \int_\Omega v_n (-\Delta)_\Omega^\alpha u dx = \int_\Omega v_n f dx. \quad (3.3)$$

Since u and v_n have the same role the above procedures, then

$$\frac{c_{N,\alpha}}{2} \int_\Omega \int_\Omega \frac{[u(x) - u(y)][v_n(x) - v_n(y)]}{|x - y|^{N+2\alpha}} dx dy = \int_\Omega u (-\Delta)_\Omega^\alpha v_n dx = \int_\Omega h_n u dx.$$

Therefore, (3.1) holds. \square

From the above observations, we are ready to prove the Integral by Parts Formula for the regional fractional Laplacian.

Theorem 3.1. *Let $u, v \in \mathbb{X}_\alpha(\Omega)$, then*

$$\int_{\Omega} \int_{\Omega} \frac{[u(x) - u(y)]^2}{|x - y|^{N+2\alpha}} dx dy \leq c_{17} \|(-\Delta)_{\Omega}^{\alpha} u\|_{L^{\infty}(\Omega)}^2 \quad (3.4)$$

and

$$\int_{\Omega} u(-\Delta)_{\Omega}^{\alpha} v dx = \frac{c_{N,\alpha}}{2} \int_{\Omega} \int_{\Omega} \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{N+2\alpha}} dx dy = \int_{\Omega} v(-\Delta)_{\Omega}^{\alpha} u dx. \quad (3.5)$$

Proof. Let $f(x) = (-\Delta)_{\Omega}^{\alpha} u(x)$, $h(x) = (-\Delta)_{\Omega}^{\alpha} v(x)$, and choose $\{f_n\}_n$, $\{h_n\}_n$ two sequences of $C^2(\Omega) \cap C(\bar{\Omega})$ functions such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{L^{\infty}(\Omega)} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|h_n - h\|_{L^{\infty}(\Omega)} = 0. \quad (3.6)$$

Let u_n and v_n be the solution of (3.2) with nonhomogeneous terms f_n and h_n respectively. Integrating (3.2) with nonhomogeneous terms f_n by u_n and v_n over Ω , from (3.3), we have that

$$\frac{c_{N,\alpha}}{2} \int_{\Omega} \int_{\Omega} \frac{[u_n(x) - u_n(y)]^2}{|x - y|^{N+2\alpha}} dx dy = \int_{\Omega} u_n(-\Delta)_{\Omega}^{\alpha} u_n dx = \int_{\Omega} u_n f_n dx \quad (3.7)$$

and

$$\frac{c_{N,\alpha}}{2} \int_{\Omega} \int_{\Omega} \frac{[u_n(x) - u_n(y)][v_n(x) - v_n(y)]}{|x - y|^{N+2\alpha}} dx dy = \int_{\Omega} v_n(-\Delta)_{\Omega}^{\alpha} u_n dx = \int_{\Omega} v_n f_n dx. \quad (3.8)$$

Since

$$\|u_n\|_{C^{\beta}(\bar{\Omega})} \leq c_{18} \|f_n\|_{L^{\infty}(\Omega)} \leq c_{39} \|(-\Delta)_{\Omega}^{\alpha} u\|_{L^{\infty}(\Omega)},$$

it infers from (3.7) that

$$\int_{\Omega} \int_{\Omega} \frac{[u_n(x) - u_n(y)]^2}{|x - y|^{N+2\alpha}} dx dy \leq c_{40} \|(-\Delta)_{\Omega}^{\alpha} u\|_{L^{\infty}(\Omega)}^2.$$

This implies that for any $\epsilon > 0$ and any $n \in \mathbb{N}$,

$$\int_{\Omega} \int_{\Omega} \frac{[u_n(x) - u_n(y)]^2}{|x - y|^{N+2\alpha}} \chi_{(\epsilon, \infty)}(|x - y|) dx dy \leq c_{41} \|(-\Delta)_{\Omega}^{\alpha} u\|_{L^{\infty}(\Omega)}^2,$$

passing to the limit as $n \rightarrow \infty$, then we obtain that for any $\epsilon > 0$,

$$\int_{\Omega} \int_{\Omega} \frac{[u(x) - u(y)]^2}{|x - y|^{N+2\alpha}} \chi_{(\epsilon, \infty)}(|x - y|) dx dy \leq c_{41} \|(-\Delta)_{\Omega}^{\alpha} u\|_{L^{\infty}(\Omega)}^2.$$

Since the left hand side of above inequality is decreasing with respect to $\epsilon > 0$ and the right hand side is independent of ϵ , so passing to the limit as $\epsilon \rightarrow 0^+$, we derive (3.4).

To prove (3.5). It is obvious that v verifies (3.4). Then $\frac{u_n(x) - u_n(y)}{|x - y|^{\frac{N+2\alpha}{2}}}$ converges to $\frac{u(x) - u(y)}{|x - y|^{\frac{N+2\alpha}{2}}}$ in $L^2(\Omega \times \Omega)$ and $\frac{v_n(x) - v_n(y)}{|x - y|^{\frac{N+2\alpha}{2}}}$ converges to $\frac{v(x) - v(y)}{|x - y|^{\frac{N+2\alpha}{2}}}$ in $L^2(\Omega \times \Omega)$, thus,

$$\lim_{n \rightarrow \infty} \int_{\Omega} \int_{\Omega} \frac{[u_n(x) - u_n(y)][v_n(x) - v_n(y)]}{|x - y|^{N+2\alpha}} dx dy = \int_{\Omega} \int_{\Omega} \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{N+2\alpha}} dx dy,$$

which, together with (3.6), implies that

$$\int_{\Omega} v(-\Delta)_{\Omega}^{\alpha} u dx = \frac{c_{N,\alpha}}{2} \int_{\Omega} \int_{\Omega} \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{N+2\alpha}} dx dy.$$

The same to conclude that

$$\int_{\Omega} u(-\Delta)_{\Omega}^{\alpha} v dx = \frac{c_{N,\alpha}}{2} \int_{\Omega} \int_{\Omega} \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{N+2\alpha}} dx dy$$

and (3.5) holds. \square

3.2. Weak solution when $f \in L^2(\Omega)$. Our aim in this subsection is to consider the weak solution of (1.1) when the nonhomogeneous term f is weakened from $L^\infty(\Omega)$ to $L^2(\Omega)$. To this end, we have to involve the fractional Hilbert space $H_0^\alpha(\Omega)$, which is the closure of $C_c^2(\Omega)$ under the norm of

$$\|u\|_{H^\alpha(\Omega)} := \left(\frac{c_{N,\alpha}}{2} \int_\Omega \int_\Omega \frac{[u(x) - u(y)]^2}{|x - y|^{N+2\alpha}} dx dy \right)^{\frac{1}{2}} + \|u\|_{L^2(\Omega)}. \quad (3.9)$$

This is called as Gagliardo norm and we denote by $\|u\|_{H_0^\alpha(\Omega)}$ the first part of (3.9) on the right hand side, which, we shall prove, is a equivalent norm of $\|u\|_{H^\alpha(\Omega)}$ in $H_0^\alpha(\Omega)$. Then we may say that the space $H_0^\alpha(\Omega)$ is the closure of $C_c^2(\bar{\Omega})$ under the norm $\|\cdot\|_{H_0^\alpha(\Omega)}$.

We make use of a Poincaré type inequality to prove the equivalence of the norms $\|\cdot\|_{H^\alpha(\Omega)}$ and $\|\cdot\|_{H_0^\alpha(\Omega)}$.

Proposition 3.2. *The norms $\|\cdot\|_{H^\alpha(\Omega)}$ and $\|\cdot\|_{H_0^\alpha(\Omega)}$ are equivalent in $H_0^\alpha(\Omega)$.*

Proof. For C^2 bounded domain and $\alpha \in (\frac{1}{2}, 1)$, it follows by [12, Theorem 1.1] that

$$\int_\Omega \frac{|u(x)|^2}{\rho^{2\alpha}(x)} dx \leq c_{41} \int_\Omega \int_\Omega \frac{[u(x) - u(y)]^2}{|x - y|^{N+2\alpha}} dx dy, \quad \forall u \in C_c^2(\Omega),$$

which implies that

$$\|u\|_{L^2(\Omega)} \leq c_{42} \|u\|_{H_0^\alpha(\Omega)}, \quad \forall u \in C_c^2(\Omega). \quad (3.10)$$

Since $C_c^2(\Omega)$ is dense in $H_0^\alpha(\Omega)$, then (3.10) holds in $H_0^\alpha(\Omega)$. We omit the left proof. \square

Proof of Theorem 1.1 part (i). Uniqueness. Let u, w be two weak solutions of (1.1), then we derive that

$$\langle u - w, v \rangle_{H_0^\alpha(\Omega)} = 0, \quad \forall v \in H_0^\alpha(\Omega).$$

Taking $v = u - w \in H_0^\alpha(\Omega)$, we have that

$$\|u - w\|_{H_0^\alpha(\Omega)} = 0.$$

Then we obtain the uniqueness.

Existence. Let $\{f_n\}_n$ be a sequence of functions in $\mathbb{X}_\alpha(\Omega)$ satisfying

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{L^2(\Omega)} = 0.$$

Let u_n be the classical solution of (1.1) with nonhomogeneous term f_n . Then

$$\langle u_n, v \rangle_{H_0^\alpha(\Omega)} = \int_\Omega v f_n dx, \quad \forall v \in H_0^\alpha(\Omega). \quad (3.11)$$

From Theorem 3.1, Proposition 3.2 and Hölder inequality, we have that

$$\begin{aligned} \|u_n\|_{H_0^\alpha(\Omega)}^2 &= \int_\Omega f_n u_n dx \leq \|u_n\|_{L^2(\Omega)} \|f_n\|_{L^2(\Omega)} \\ &\leq c_{43} \|u_n\|_{H_0^\alpha(\Omega)} \|f_n\|_{L^2(\Omega)}. \end{aligned}$$

Then we have that

$$\|u_n\|_{H_0^\alpha(\Omega)} \leq c_{43} \|f_n\|_{L^2(\Omega)} \leq c_{44} \|f\|_{L^2(\Omega)}. \quad (3.12)$$

From [10, Theorem 6.7, Theorem 7.1], the embedding: $H_0^\alpha(\Omega) \hookrightarrow L^2(\Omega)$ is compact, then up to subsequence, there exists $u \in L^2(\Omega)$ such that

$$u_n \rightarrow u \text{ in } (H_0^\alpha(\Omega))' \text{ as } n \rightarrow \infty$$

and

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{L^2(\Omega)} = 0.$$

Then from (3.11), we have that

$$\langle u, v \rangle_{H_0^\alpha(\Omega)} = \int_{\Omega} v f \, dx, \quad \forall v \in H_0^\alpha(\Omega),$$

that is, u is a weak solution of (1.1). Taking $v = u$ above, we deduce (1.2). \square

4. VERY WEAK SOLUTION

4.1. The case that $f \in L^1(\Omega, \rho^\beta \, dx)$. In this section, we may weaken the nonhomogeneous term f to $L^1(\Omega, \rho^\beta \, dx)$ in (1.1).

Proof of Theorem 1.1 part (ii) when $f \in L^1(\Omega, \rho^\beta \, dx)$. *Uniqueness.* Let u, w be two very weak solutions of (1.1), then

$$\int_{\Omega} (u - w)(-\Delta)_{\Omega}^{\alpha} v \, dx = 0, \quad \forall v \in \mathbb{X}_{\alpha}(\Omega). \quad (4.1)$$

Let η_{u-w} be the solution of

$$\begin{cases} (-\Delta)_{\Omega}^{\alpha} \eta_{u-w} = \text{sign}(u - w) & \text{in } \Omega, \\ \eta_{u-w} = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.2)$$

We observe that $\eta_{u-w} \in \mathbb{X}_{\alpha}(\Omega)$ and put $v = \eta_{u-w}$ in (4.1), then we obtain that

$$\int_{\Omega} |u - w| \, dx = 0,$$

which implies the uniqueness.

Existence. We make use of Proposition 3.1 to approximate the solution u of (1.1) by a sequence of classical solutions. In fact, we choose a sequence of $C^2(\Omega) \cap C(\bar{\Omega})$ functions $\{f_n\}_n$ such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{L^1(\Omega, \rho^\beta \, dx)} = 0. \quad (4.3)$$

Denote u_n the solution of (1.1) with nonhomogeneous nonlinearity f_n . Then from Proposition 3.1, we have that

$$\int_{\Omega} u_n (-\Delta)_{\Omega}^{\alpha} v \, dx = \int_{\Omega} f_n(x) v(x) \, dx, \quad \forall v \in \mathbb{X}_{\alpha}(\Omega). \quad (4.4)$$

By Lemma 2.5, it deduces that

$$|v(x)| \leq c_{45} \rho^\beta(x), \quad \forall x \in \Omega \quad (4.5)$$

and together with the convergence of $\{f_n\}_n$ in $L^1(\Omega, \rho^\beta \, dx)$, we obtain that

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n(x) v(x) \, dx = \int_{\Omega} f(x) v(x) \, dx. \quad (4.6)$$

For any $n, m \in \mathbb{N}$, let $\eta_{u_m - u_n}$ be the solution of (4.2) with nonhomogeneous term $\text{sign}(u_m - u_n)$, then we obtain that

$$\int_{\Omega} |u_m - u_n| \, dx = \int_{\Omega} (f_m - f_n) \eta_{u_m - u_n} \, dx \leq c_{46} \int_{\Omega} |f_m - f_n| \rho^\beta \, dx.$$

For any $\epsilon > 0$, it infers by (4.3) that there exists $N_\epsilon > 0$ such that for any $n, m > N_\epsilon$,

$$c_{47} \int_{\Omega} |f_m - f_n| \rho^\beta \, dx \leq \epsilon,$$

which implies that for any $n, m > N_\epsilon$

$$\int_{\Omega} |u_m - u_n| \, dx \leq \epsilon.$$

Thus, $\{u_n\}_n$ is a Cauchy sequence in $L^1(\Omega)$ and then there exists $u \in L^1(\Omega)$ such that

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{L^1(\Omega)} = 0.$$

Passing to the limit of (4.4) as $n \rightarrow \infty$, we obtain that

$$\int_{\Omega} u(x)(-\Delta)_{\Omega}^{\alpha} v(x) dx = \int_{\Omega} f(x)v(x) dx, \quad \forall v \in \mathbb{X}_{\alpha}(\Omega).$$

Therefore, problem (1.1) has a very weak solution, that is,

$$\int_{\Omega} u(-\Delta)_{\Omega}^{\alpha} v dx = \int_{\Omega} f v dx, \quad \forall v \in \mathbb{X}_{\alpha}(\Omega), \quad (4.7)$$

choosing $v = \eta_u$, the solution of (4.2) with nonhomogeneous term $\text{sign}(u)$, it infers from (4.7) that

$$\int_{\Omega} |u| dx \leq c_{48} \int_{\Omega} |f| \rho^{\beta} dx.$$

The proof ends. \square

4.2. The case that $f \in \mathcal{M}(\Omega, \rho^{\beta})$. In this subsection, we may weaken the nonhomogeneous term f to $\mathcal{M}(\Omega, \rho^{\beta})$ in (1.1).

Proof of Theorem 1.1 part (ii) when $f \in \mathcal{M}(\Omega, \rho^{\beta})$. *Uniqueness.* Let u, w be two very weak solutions of (1.1), then

$$\int_{\Omega} (u - w)(-\Delta)_{\Omega}^{\alpha} v dx = 0, \quad \forall v \in \mathbb{X}_{\alpha}(\Omega),$$

which reduces to (4.1).

Existence. We make use of Proposition 3.1 to approximate the solution u of (1.1) by a sequence of classical solutions. In fact, we choose a sequence of $C^2(\Omega) \cap C(\bar{\Omega})$ functions $\{f_n\}_n$ such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n v dx = \int_{\Omega} v df \quad \text{for any } v \in \mathbb{X}_{\alpha}(\Omega). \quad (4.8)$$

Denote u_n the solution of (1.1) with nonhomogeneous nonlinearity f_n . Then from Proposition 3.1, we have that

$$\int_{\Omega} u_n(-\Delta)_{\Omega}^{\alpha} v dx = \int_{\Omega} f_n(x)v(x) dx, \quad \forall v \in \mathbb{X}_{\alpha}(\Omega). \quad (4.9)$$

Thus, it deduces by (4.5) and (4.8) that

$$\lim_{n \rightarrow \infty} \int_{\Omega} v(x)f_n(x) dx = \int_{\Omega} v(x) df(x). \quad (4.10)$$

To prove that $\{u_n\}_n$ is uniformly bounded in $L^1(\Omega)$. For any $n \in \mathbb{N}$, let η_{u_n} be the solution of (4.2) with nonhomogeneous term $\text{sign}(u_n)$, then we derive that

$$\begin{aligned} \int_{\Omega} |u_n| dx &= \int_{\Omega} f_n \eta_{u_n} dx \leq c_{49} \int_{\Omega} |\eta_{u_n}| d|f| \\ &\leq c_{49} c_{48} \|\eta_{u_n} \rho^{-\beta}\|_{L^{\infty}(\Omega)} \int_{\Omega} \rho^{\beta} d|f| \leq c_{50} \int_{\Omega} \rho^{\beta} d|f|, \end{aligned}$$

where $c_{49}, c_{50} > 0$ are independent of n , since $|\eta_{u_n}| \leq c_{48} \rho^{\beta}$ in Ω .

To prove that $\{u_n\}_n$ is uniformly integrable. Let \mathcal{O} be any Borel subset of Ω , take $\eta_{\mathcal{O}}$ be the solution of (4.2) with nonhomogeneous term $\chi_{\mathcal{O}}\text{sign}(\mathcal{O})$, then we see that

$$\begin{aligned} \int_{\mathcal{O}} |u_n| dx &= \int_{\Omega} f_n \eta_{\mathcal{O}} dx \leq c_{49} \int_{\Omega} |\eta_{\mathcal{O}}| d|f| \\ &\leq c_{49} c_{48} \|\eta_{\mathcal{O}} \rho^{-\beta}\|_{L^\infty(\Omega)} \int_{\Omega} \rho^\beta d|f| \leq c_{51} \int_{\Omega} \rho^\beta d|f|, \end{aligned}$$

where $c_{51} > 0$ is independent of n . We observe that

$$\begin{aligned} |\rho^{-\beta}(x) \eta_{\mathcal{O}}(x)| &= \rho^{-\beta}(x) \left| \int_{\Omega} G_{\Omega, \alpha}(x, y) \chi_{\mathcal{O}}(y) \text{sign}(u_n)(y) dy \right| \\ &\leq c_7 \int_{\mathcal{O}} \frac{\rho^\beta(y)}{|x - y|^{N-2+2\alpha}} dy \\ &\leq c_7 D_0^\beta \int_{B_{d_0}(x)} \frac{1}{|x - y|^{N-2+2\alpha}} dy \\ &\leq c_{52} d_0^{2-2\alpha} = c_{52} |\mathcal{O}|^{\frac{2-2\alpha}{N}}, \end{aligned}$$

where $c_{52} > 0$ is independent of n , $D_0 = \sup_{x, y \in \omega} |x - y|$ and $d_0 > 0$ satisfying

$$|\mathcal{O}| = |B_{d_0}(0)|.$$

Thus, we derive that

$$\int_{\mathcal{O}} |u_n| dx \leq c_{53} \|f\|_{\mathcal{M}_\alpha(\Omega)} |\mathcal{O}|^{\frac{2-2\alpha}{N}},$$

where $c_{53} > 0$ is independent of n .

Therefore, we conclude that $\{u_n\}_n$ is uniformly bounded in $L^1(\Omega)$ and uniformly integrable, thus weakly compact in $L^1(\Omega)$ by the Dunford-Pettis Theorem, and there exists a subsequence $\{u_{n_k}\}_k$ and an integrable function u such that $u_{n_k} \rightarrow u$ weakly in $L^1(\Omega)$. Passing to the limit in (4.9), we obtain that u is a very weak solution of (1.1).

Choosing $v = \eta_u$, the solution of (4.2) with nonhomogeneous term $\text{sign}(u)$, it infers that

$$\int_{\Omega} |u| dx \leq c_{54} \int_{\Omega} \rho^\beta d|f|.$$

This ends the proof. \square

5. GENERAL BOUNDARY DATA

In this section, we consider the classical solution of (1.5) under the general boundary data. In [15, Theorem 1.3 (i)], the author proved the Integral by Part Formula

$$\int_{\Omega} u(-\Delta)_{\Omega}^{\alpha} v dx = \int_{\Omega} v(-\Delta)_{\Omega}^{\alpha} u dx + \int_{\partial\Omega} v \frac{\partial^{\beta} u}{\partial \bar{n}^{\beta}} d\omega - \int_{\partial\Omega} u \frac{\partial^{\beta} v}{\partial \bar{n}^{\beta}} d\omega, \quad \forall u, v \in \mathbb{D}_{\beta}, \quad (5.1)$$

where \mathbb{D}_{β} is given by (1.7).

However, it is open to show that the solution $u_{f,g}$ of (1.5) belongs to \mathbb{D}_{β} , even under the hypothesis that

$$f \in C^2(\Omega) \cap C(\bar{\Omega}) \quad \text{and} \quad g \in C^2(\partial\Omega).$$

Proposition 5.1. *Assume that $f \in C^2(\Omega) \cap C(\bar{\Omega})$, $g \in C^2(\partial\Omega)$ and u be the classical solution of (1.5). Then*

$$\int_{\Omega} u(-\Delta)_{\Omega}^{\alpha} v dx = \int_{\Omega} f v dx + \int_{\partial\Omega} g \frac{\partial^{\beta} v}{\partial \bar{n}^{\beta}} d\omega, \quad \forall v \in \mathbb{X}_{\alpha}(\Omega) \cap \mathbb{D}_{\beta}. \quad (5.2)$$

Proof. Since Ω is a C^2 domain and $g \in C^2(\partial\Omega)$, then there exists a $C^2(\bar{\Omega})$ function G such that

$$G = g \quad \text{on} \quad \partial\Omega.$$

Let u_g be the solution of (1.5) and denote

$$u_0 = u_g - G.$$

Then u_0 satisfies (2.15). From [16, Proposition 2.3], $(-\Delta)_\Omega^\alpha G \in C_{loc}^{2-2\alpha}(\Omega)$ and

$$|(-\Delta)_\Omega^\alpha G(x)| \leq c_{55} \rho(x)^{-\beta}, \quad \forall x \in \Omega. \quad (5.3)$$

Choose \tilde{g}_n a sequence of $C^2(\Omega) \cap C(\bar{\Omega})$ functions such that

$$\lim_{n \rightarrow \infty} \|(\tilde{g}_n - (-\Delta)_\Omega^\alpha G) \rho^\beta\|_{L^\infty(\Omega)} = 0.$$

Let w_n be the solution of

$$\begin{cases} (-\Delta)_\Omega^\alpha u = f - \tilde{g}_n & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.4)$$

By Proposition 3.1, it infers that for $v \in \mathbb{X}_\alpha(\Omega)$,

$$\begin{aligned} \int_\Omega w_n(x) (-\Delta)_\Omega^\alpha v(x) dx &= \int_\Omega v(x) (-\Delta)_\Omega^\alpha w_n(x) dx \\ &= \int_\Omega v(x) f(x) dx - \int_\Omega v(x) \tilde{g}_n dx. \end{aligned} \quad (5.5)$$

We observe that

$$\|w_n - u_g\|_{L^\infty(\Omega)} \leq c_{37} \|(\tilde{g}_n - (-\Delta)_\Omega^\alpha G) \rho^\beta\|_{L^\infty(\Omega)}.$$

Therefore, passing to the limit of (5.5) as $n \rightarrow \infty$, we have that

$$\int_\Omega u_0(x) (-\Delta)_\Omega^\alpha v(x) dx = \int_\Omega v(x) f(x) dx - \int_\Omega v(x) (-\Delta)_\Omega^\alpha G(x) dx.$$

Since $G \in C^2(\bar{\Omega})$, then for $v \in \mathbb{D}_\beta$, it deduces by (5.1) that

$$\int_\Omega v(x) (-\Delta)_\Omega^\alpha G(x) dx = \int_\Omega G(x) (-\Delta)_\Omega^\alpha v(x) dx - \int_{\partial\Omega} g \frac{\partial^\beta v}{\partial \vec{n}^\beta} d\omega.$$

Thus,

$$\int_\Omega u_g (-\Delta)_\Omega^\alpha v dx = \int_\Omega (u_0 + G) (-\Delta)_\Omega^\alpha v dx = \int_\Omega v f dx + \int_{\partial\Omega} g \frac{\partial^\beta v}{\partial \vec{n}^\beta} d\omega.$$

The proof ends. \square

Remark 5.1. The function space $C_c^2(\Omega)$ is a subset of $\mathbb{X}_\alpha(\Omega) \cap \mathbb{D}_\beta$, so $\mathbb{X}_\alpha(\Omega) \cap \mathbb{D}_\beta$ is not empty. However, $C_c^2(\Omega)$ is not proper for the Integral by Parts Formula since the boundary term $\int_{\partial\Omega} g \frac{\partial^\beta v}{\partial \vec{n}^\beta} d\omega$ always vanishes for $v \in C_c^2(\Omega)$.

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